Physica Macedonica **61**, (2012) p. 85-89
ISSN 1409-7168

WOODS-SAXON POTENTIAL IN THE PRESENCE OF A COSMIC STRING

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Abstract. We study the behavior of a non-relativistic quantum system of a neutron in a mean-field Woods-Saxon potential in the presence of cosmic string. The corresponding Schrödinger equation with this potential has been solved by modification of the radial part to become comparable with the associated Jacobi differential equation. We find the energy spectra and wave functions for this system and explore some differences compared to the analog system in flat Minkowski space-time.

PACS: 21.60.Ev, 04.20.Gz, 11.25.-w

1. INTRODUCTION

The effects of the gravitational field within the atomic realm continues to be an interesting research topic, with wide range of, relativistic and nonrelativistic, energies and scales. Especially important, in this context, are potentially observable changes in the energy levels of the microscopic quantum systems, due to geometric curvature, and / or other topological features of the space-time at the location of the system. One such a feature is the presence of a cosmic string, manifested as a topological defect, at atomic level.

Cosmic strings are exotic topological defects, suggested as an explanation for phase transitions in the very early Universe and of the observed global distribution of mass in galaxies, with spontaneously braking of extreme smoothness of the early space-time as their main mechanism of physical infuence [1]. As of today, however there is no experimental evidence that support their actual existence in nature and their use in theoretical models are of pure speculative nature. If exist they may have a word in some explanations of ultrahigh energy neutrinos, cosmological gamma-ray burst as well as the anisotropy of the cosmic microwave background radiation too. Their geometry should be localy flat, and may be described as conical which means that a particle near them does not feel local gravity. However they should change the energy levels of quantum systems due to the non-trivial topology of space-time around them [1,2].

In a recent study, Geusa de A. Marques and Valdir B. Bezerra [3,4 and references therein] have shown that an atom in a gravitational field is influenced by its interaction with the local curvature, as well as with the topology of the space-time, and as a consequence there is a shift in the energies at the atomic level. The aim of this paper is to extend their model to nuclei, within the framework of mean-field Woods-Saxon potential. We have used the approach of M. R. Pahlavani, J. Sadeghi and M. Ghezelbash [5,6] were the radial Schrödinger equation, for small r, is solved by comparison with the associated Jacobi differential equation, and closed formula for energy levels are found, for this potential.

2. FORMULATION OF THE PROBLEM

To find the metric of the space-time one needs to solve the Einstein field equation

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=8\pi GT_{\mu\nu}, \tag{1}$$

where $R_{\mu\nu}$ is Ricci tensor, $g_{\mu\nu}$ is the metric tensor, R the Ricci scalar and $T_{\mu\nu}$ is the stress-energy tensor (as for the unit system we adopt $c=1$). One can show (see, for details [1,2]) that the exterior metric of an infinitely long straight and static cosmic string in spherical coordinates, takes the form

$$ds^2=-dt^2+dr^2+r^2d\theta^2+\alpha^2r^2\sin^2\theta d\phi^2, \tag{2}$$

where $0<r<\infty$, $0<\theta<\pi$, $0<\varphi\leq 2\pi$ and $\alpha=1-8\pi G\mu$ belongs to the interval $(0,1]$; μ being the linear mass density of the cosmic string. Thus, the space-time produced by an idealized cosmic string is locally flat, however globally may be described by conical geometry, with a planar angle deficit determined by $\delta=8\pi G\mu$.

So, we need to solve the Shrödinger equaion in the presence of cosmic string, given by

$$i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2_{LB}\Psi+V\Psi\ . \tag{3}$$

where the ∇_{LB} is the Laplace-Beltrami operator on the Riemannian manifold with the corresponding metric [7] and, in this case reads

$$\nabla^2_{LB}=\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{\tan\theta}\frac{\partial}{\partial\theta}+\frac{\partial^2}{\partial\theta^2}+\frac{1}{\alpha^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right], \tag{4}$$

and V is the main-field Woods-Saxon potential

$$V(r)=-\frac{V_0}{1+e^{\frac{r-R_0}{a}}}, \tag{5}$$

R_0 and a being geometrical characteristics of the nuclei, and V_0 is the depth of the potential well at $r=0$ [8]. With this we establish the theoretical context of our problem and proceed to its solution, in the following section, in units $\hbar=1, m=1$.

3. SOLUTION OF THE SCHRÖDINGER EQUATION

After transition to time-independent Schrödinger equation for ψ, in stationary environment, and substition of the expression for the potential, $V(r)$ one may proceed to standard separation of variables with the ansatz $\psi(r,\theta,\varphi) = R(r)\Theta(\theta)\Phi(\varphi)$. To simplify the expressions further we introduce new variables for the Woods-Saxon potential $r - R_0 \to r$ and $1/a \to 2\omega$.

The azimuthal part being the simplest to solve, $\Phi(\varphi) = C\exp(im\varphi)$. C is a constant and m assumes the values $m = 0,\pm1,\pm2,...$ while for the other two parts we get

$$\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)+2\left(E+\frac{V_0}{1+e^{2\omega r}}\right)-\frac{\lambda}{r^2}\right]R(r)=0, \tag{6}$$

$$\left[\frac{1}{\tan\theta}\frac{\partial}{\partial\theta}+\frac{\partial^2}{\partial\theta^2}-\frac{m^2}{\alpha^2\sin^2\theta}+\lambda\right]\Theta(\theta)=0, \tag{7}$$

λ being the separation constant.

By making the substitution $x = \cos\theta$ in the last equation one may obtain a type of Legendre differential equation with dependence on a parameter α whose solutions can be written via the generalized associated Legendre functions $\Theta(\theta) = P_{l_{(\alpha)}}^{m_{(\alpha)}}(\cos\theta)$, (see references [3,4] for details) with $m_{(\alpha)} \equiv m/\alpha$,$l_{(\alpha)} \equiv l - (1 - 1/\alpha)m$, $l = 0,1,2,...$ and for the separation constant, $\lambda = l_{(\alpha)}(l_{(\alpha)}+1)$. Now we make an approximation, valid for $r \to 0$,

$$r=\frac{1}{2\omega}\left(e^{2\omega r}-1\right) \tag{8}$$

and solve the radial part of Schrödinger equation, by making the substitution $R(r) = u(r)/r$ and introducing a new variable $x = \tanh(\omega r)$ to obtain

$$\left(1-x^2\right)\frac{d^2u(x)}{dx^2}-2x\frac{du(x)}{dx}+\left[\frac{\varepsilon^2}{\left(1-x^2\right)}+\frac{\nu^2}{\left(1+x\right)}-\frac{l_{(\alpha)}\left(l_{(\alpha)}+1\right)}{x^2}\frac{\left(1-x\right)}{\left(1+x\right)}\right]u(x)=0, \tag{9}$$

where with $\varepsilon^2 = \left|2mE/\omega^2\right|$, $\nu^2 = V_0/\omega^2$ we follow the notation from [3]. This equation can be modified to become comparable with Jacobi differential equation with real parameters $\gamma,\beta > -1$, in the interval $x \in (-1,1)$

$$\begin{aligned}&\left(1-x^2\right)\frac{d^2}{dx^2}P_{nl}^{(\gamma,\beta)}(x)+\left(\beta-\gamma-\left(\gamma+\beta+2\right)x\right)\frac{d}{dx}P_{nl}^{(\gamma,\beta)}(x)+\\&+\left[n\left(\gamma+\beta+n+1\right)-\frac{l\left(\gamma+\beta+l+\left(\gamma-\beta\right)x\right)}{1-x^2}\right]P_{nl}^{(\gamma,\beta)}(x)=0\end{aligned} \tag{10}$$

where n,l are non-negative integers and $0 \le l \le n < \infty$.

Let us substitute $u(x)$ to be product of two, tentatively unknown functions $v(x)$ and $w(x)$ to obtain

$$\begin{aligned}&\left(1-x^2\right)\frac{d^2}{dx^2}v(x)+\left[2\left(1-x^2\right)\frac{w'(x)}{w(x)}-2x\right]\frac{dv(x)}{dx}+\\ &+\left[\left(1-x^2\right)\frac{w''(x)}{w(x)}-2x\frac{w'(x)}{w(x)}+\frac{\varepsilon^2}{\left(1-x^2\right)}+\frac{\nu^2}{\left(1+x\right)}-\frac{l_{(\alpha)}\left(l_{(\alpha)}+1\right)}{x^2}\frac{\left(1-x\right)}{\left(1+x\right)}\right]v(x)=0\end{aligned} \tag{11}$$

By comparing the last two equations, one may easily find $w(x)=C(1+x)^{\beta/2}(1-x)^{\gamma/2}$, C being some constant. Similarly, one may find that from this comparision follows

$$\left(\frac{\gamma^2}{4}-\frac{\gamma}{2}\right)+\left(\frac{\beta^2}{4}-\frac{\beta}{2}\right)-\frac{\gamma\beta}{2}+\varepsilon^2+\nu^2-l_{(\alpha)}\left(l_{(\alpha)}+1\right)=n\left(\gamma+\beta+n+1\right)-l\left(\gamma+\beta+l\right)$$

$$2\left(\frac{\gamma^2}{4}-\frac{\gamma}{2}\right)-2\left(\frac{\beta^2}{4}-\frac{\beta}{2}\right)+\gamma-\beta-\nu^2=-l\left(\gamma-\beta\right)$$

$$\left(\frac{\gamma^2}{4}-\frac{\gamma}{2}\right)+\left(\frac{\beta^2}{4}-\frac{\beta}{2}\right)+\frac{\gamma\beta}{2}+\gamma+\beta=-n\left(\gamma+\beta+n+1\right)$$

$$l_{(\alpha)}\left(l_{(\alpha)}+1\right)=0$$

From these four equations one may, with usual algebraic manipulations, find expression for ε^2 and for energy, as following

$$E_{nl_{(\alpha)}}=-\frac{\hbar^2\omega^2}{2m}\left(\gamma+l\right)^2, \tag{12}$$

provided the quantization condition for α , from $l_{(\alpha)}(l_{(\alpha)}+1)=0$ is fulfilled; hence the index $l_{(\alpha)}$. Putting together all parts of the wave function, we arrive at

$$\begin{aligned}&\Psi_{nlm\alpha}(r,\theta,\phi)=\\ &=\frac{C}{r}\left(1+\tanh\omega r\right)^{\beta/2}\left(1-\tanh\omega r\right)^{\gamma/2}e^{im\phi}\,P_{l_{(\alpha)}}^{m_{(\alpha)}}(\cos\theta)P_l^{(\gamma,\beta)}(\tanh\omega r).\end{aligned} \tag{13}$$

4. CONCLUSIONS

The non-trivial space-time background, in general shifts the energy levels of quantum systems and in some simple potentials with high symmetry, can restrict even their existence with making additional conditions for the theoretical framework to be fulfilled, such as above $l_{(\alpha)}\left(l_{(\alpha)}+1\right)=0$. This conclusion, supported in this paper, for mean-field Woods-Saxon potential without a Coulomb barrier, agree in full with the results presented in available references for other simple potentials.

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